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Gravitational and sound waves in stiff matter

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Abstract. The dynamics of a perfect fluid with pressure equal to energy density is considered in space–times with planar, pseudoplanar, cylindrical and toroidal symmetries. A scheme for obtaining the general exact solution of the field equations is described. It is shown that this scheme admits inclusion of additional massless, minimally coupled scalar fields and (for planar symmetry) electromagnetic plane waves. Two physical examples are discussed: (i) finite perturbations of a static regular cylinder and (ii) waves of finite magnitude in a Kasner-like universe. In case (i) it is shown that all physically allowed perturbations preserving cylindrical symmetry are of standing-wave type and cannot propagate. Thus a perturbed static stiff cylinder cannot radiate the Einstein–Rosen waves. In case (ii) it is shown that travelling-wave perturbations obey a sort of momentum-conservation law; besides, it is argued that sound perturbations can provide a mechanism for fragmentation in a universe filled with stiff matter.

1. Introduction

Knowledge of exact solutions of general relativity involving waves within matter is highly desirable. Solutions of this type could contribute to our understanding of such processes as, e.g., interaction between gravitational waves and matter and radiative gravitational collapse.

Most existing wave solutions describe vacuum or electrovacuum space–times (see Zakharov 1972). Problems involving matter are in general very complicated, since one has to deal with interaction between the gravitational waves and the sound waves excited by them. The simplest case is that of 'stiff matter' (a perfect fluid with equal pressure and energy density, $p=\rho$), because in such matter the velocity of sound equals that of light. The equation of state $p=\rho$ corresponds to maximum stiffness compatible with causality, and is perhaps a good approximation for supernuclear densities of matter (Zeldovich and Novikov 1971, Staniukovich *et al* 1975).

In this paper, the dynamics of a $p=\rho$ fluid is considered in space–times with the line element

$$ds^2 = e^{2\gamma}\, dT^2 - e^{2\lambda}\, dR^2 - a^2(e^{2\omega}\, d\xi^2 + e^{-2\omega}\, d\eta^2) \tag{1}$$

where γ, λ, a and ω are functions of R and T. This metric corresponds to pseudoplanar, toroidal or cylindrical symmetry when the coordinate lines of ξ and η (Killing vector orbits) are assumed to be, respectively, both open, both closed or one open (say, η) and one closed; in the latter case $x^2=\xi$ is the azimuthal coordinate and $x^3=\eta$ the longitudinal one. The function $a(R, T)$ may be called a scale factor and $\omega(R, T)$ an anisotropy factor at the (ξ, η) surfaces. Planar symmetry is a special case ($\omega=0$) of the pseudoplanar one. The field equations involve only local quantities and may be treated for all these symmetries simultaneously.

In § 2 the problem is treated in its general form. Using the scalar field representation (Tabensky and Taub 1973) for stiff matter and the coordinate condition $\gamma \equiv \lambda$, we reduce the problem to that for a set of scalar fields in a plane-symmetric space–time, i.e. essentially to that solved by Tabensky and Zamorano (1975). Thus a general solution is available for stiff matter dynamics in space–times with metric (1). Moreover, one can quite easily include an additional massless scalar field (or a multiplet of these), if required.

Then we discuss two physical models with metric (1). Namely, in § 3 we consider finite perturbations of a non-singular static cylinder of stiff matter and show that it cannot radiate waves of the Einstein–Rosen type. The reason is that travelling waves can exist only if there is a singularity at the axis. In § 4, we show that in a Kasner-like universe travelling waves cause strong inhomogeneities, and argue that a certain type of perturbations (sound ones) can cause fragmentation.

In appendix 1, it is shown that for $\omega = 0$ (planar symmetry condition) the scheme of § 2 admits inclusion of plane electromagnetic waves travelling in the R direction. In appendix 2 the problem is discussed in the co-moving frame of reference. It is shown that a broad, though not general, class of solutions can be obtained in this frame directly.

Various special cases of the problem have been considered by Tabensky and Taub (1973), Letelier (1975), Letelier and Tabensky (1975a, b), Charach and Malin (1979, see also references therein). Letelier and Tabensky discussed the general character of the singularities emerging in the solutions with stiff matter. Charach and Malin considered the solution for gravitational and scalar waves in a matter-free cosmology corresponding to our case (i) (see § 2). They also treated a high-frequency limit for these types of waves. This led to some new solutions describing null fluid flows. Evidently the solutions considered here can be modified in the same way, but this goes beyond the frame of this paper. Finally, Singh and Yadav (1978) obtained some special solutions for a more general class of space–times (with $g_{23} \neq 0$).

2. Solution of field equations

A non-rotating perfect fluid with $p = \rho$ and 4-velocity u^μ can be represented by an auxiliary scalar field σ (Tabensky and Taub 1973):

$$\rho = p = \sigma^{,\alpha}\sigma_{,\alpha}, \qquad u^\mu = \sigma^{,\mu}(\sigma^{,\alpha}\sigma_{,\alpha})^{-1/2}, \qquad \nabla^\alpha\nabla_\alpha\sigma = 0. \tag{2}$$

The fluid energy–momentum tensor becomes that for the scalar field,

$$T^\nu_\mu = (2u_\mu u^\nu - \delta^\nu_\mu)p = 2\sigma_{,\mu}\sigma^{,\nu} - \delta^\nu_\mu\sigma^{,\alpha}\sigma_{,\alpha}. \tag{3}$$

Consequently, one can write the field equations of general relativity for metric (1) and $\sigma = \sigma(R, T)$ in the form

$$(\mathrm{e}^{\gamma-\lambda}aa')' = (\mathrm{e}^{\lambda-\gamma}a\dot{a})^{\cdot}, \tag{4}$$

$$\dot{a}' - \gamma'\dot{a} - \dot{\lambda}a' + a\dot{\omega}^A\omega'^A = 0, \tag{5}$$

$$\mathrm{e}^{-2\lambda}(a'^2 + 2\gamma'aa' - a^2\omega'^A\omega'^A) - \mathrm{e}^{-2\gamma}(2a\ddot{a} + \dot{a}^2 - 2\dot{\gamma}a\dot{a} + a^2\dot{\omega}^A\dot{\omega}^A) = 0, \tag{6}$$

$$(\mathrm{e}^{\gamma-\lambda}a^2\omega'^A)' = (\mathrm{e}^{\lambda-\gamma}a^2\dot{\omega}^A)^{\cdot}. \tag{7}$$

Here dots and primes denote $\partial/\partial T$ and $\partial/\partial R$ respectively and ω^A are the effective scalar fields:

$$\omega^1 \overset{\text{def}}{=} \omega, \qquad \omega^2 \overset{\text{def}}{=} \kappa^{1/2}\sigma. \tag{8}$$

κ is the gravitational constant and summation over repeated A is meant. Equations (4)–(6) are the Einstein equations $\binom{0}{0}+\binom{1}{1}$, (01) and $\binom{1}{1}$, respectively, while (7) for $A=1$ is the $\binom{2}{2}-\binom{3}{3}$ Einstein equation and for $A=2$ the scalar equation (2). Thus both the fluid and the anisotropy factor behave like massless, minimally coupled, source-free scalar fields. If one includes additional scalar fields ψ^A of the same kind with the Lagrangians $\psi^{A,\alpha}\psi^A_{,\alpha}$, then there exist more $\omega^A = \kappa^{1/2}\psi^A (A=3, 4, \ldots)$ satisfying equation (7).

Imposing the coordinate condition $\gamma \equiv \lambda$, one easily solves (4):

$$a^2 = f(T+R) + g(T-R). \tag{9}$$

However, further on one need not deal with this general solution: similarly to the famous paper by Einstein and Rosen (1937), one can employ invariance of the condition $\gamma \equiv \lambda$ under transformations $(T, R) \to (\bar{T}, \bar{R})$ such that $\ddot{\bar{T}} = \bar{T}''$ and $\ddot{\bar{R}} = \bar{R}''$. Consequently, in the three physically different cases when the gradient $a_{,\mu}$ is timelike, spacelike or null†, one can choose the coordinates so that, respectively, (i) $a^2 = T$, (ii) $a^2 = R$, and (iii) $a^2 = (T-R)^2$.

In cases (i) and (ii) the wave equations (7) take the forms

$$\text{(i)}\ \omega''^A = T^{-1}(T\dot{\omega}^A)^{\cdot}, \qquad \text{(ii)}\ R^{-1}(R\omega'^A)' = \ddot{\omega}^A. \tag{10}$$

These equations are solved in terms of zero-order cylindrical functions by standard separation of variables or in an integral form (Tabensky and Zamorano 1975). Now equations (5) and (6) express $\dot{\gamma}$ and γ' in terms of known functions:

$$\text{(i)} \qquad \gamma' = 2T\omega'^A\dot{\omega}^A, \qquad \dot{\gamma} = -1/4T + T(\dot{\omega}^A\dot{\omega}^A + \omega'^A\omega'^A), \tag{11}$$

$$\text{(ii)} \qquad \dot{\gamma} = 2R\omega'^A\dot{\omega}^A, \qquad \gamma' = -1/4R + R(\dot{\omega}^A\dot{\omega}^A + \omega'^A\omega'^A). \tag{12}$$

These equations are easily integrated, thus completing the solution scheme. The integrability condition $\dot{\gamma}' = \gamma'^{\cdot}$ is satisfied automatically, due to (7).

In case (i), the scale factor a (which is an analogue of r, the curvature radius of coordinate spheres in spherically symmetric space–times) is timelike, and the solution may be naturally called a T-region (cosmological type) solution. In case (ii) we have then an R-region solution.

In case (iii), it is convenient to use the null coordinates $u = T+R$ and $v = T-R$. Equations (5) and (6) lead to

$$2\gamma_v = v\omega^A_v\omega^A_v, \qquad \omega^A_u\omega^A_u = 0, \tag{13}$$

where the indices 'u' and 'v' stand for $\partial/\partial u$ and $\partial/\partial v$, respectively. Thus $\omega^A = \omega^A(v)$ are arbitrary functions of one variable (equations (7) are now satisfied automatically) and

$$\gamma = \Gamma_1(u) + \Gamma_2(v), \qquad 2\Gamma_2 = \int \mathrm{d}v\, v\omega^A_v\omega^A_v, \tag{14}$$

† The possibility a = constant is rejected since it brings equation (6) to the form $\omega'^A\omega'^A + \dot{\omega}^A\dot{\omega}^A = 0$, whence ω^A = constant. However, if among ω^A there is a so-called repulsive scalar field (that with a minus sign before the Lagrangian (see e.g. Wagoner 1970)), then in the summation in A the corresponding term is negative and equation (6) just connects different ω^A, each of these obeying the equation $\omega'' = \ddot{\omega}$. In this case, the remaining unknown function γ is found from the equation $\ddot{\gamma} - \gamma'' + \omega^{A,\alpha}\omega^A_{,\alpha} = 0$. This is the $\binom{2}{3}+\binom{2}{3}$ Einstein equation, which was omitted in the initial set because in the case $a \neq$ constant it is a consequence of equations (4)–(7).

with an arbitrary function $\Gamma_1(u)$ which can be absorbed by a coordinate transformation $u \to u' = f(u)$. This is a special solution, which for the case of one scalar field ω has been discussed in some detail by Tabensky and Zamorano (1975). However, this is a purely scalar-vacuum solution: $p = \sigma^{,\sigma}\sigma_{,\alpha} = 0$.

Thus the general solution for stiff matter reduces to cases (i) and (ii).

In §§ 3 and 4, the general solution is applied to treat finite perturbations of two known background configurations characterised by certain $\omega = \omega_0$ and $\sigma = \sigma_0$. The perturbations are naturally classified as matter (sound) and gravitational (anisotropy) ones†; they are characterised, respectively, by $\sigma - \sigma_0 \neq 0$ and $\omega - \omega_0 \neq 0$. These two types of perturbations will be considered separately, as they contribute to the function $\gamma(R, T)$ quite independently and their joint existence causes no new essential effects.

3. Finite perturbations of a static cylinder

A singularity-free, static, cylindrically symmetric solution for stiff matter is obtained from § 2 in the special case $a^2 = R$, $\omega = \omega(R)$ and $\sigma = \sigma(T)$, and may be written in the form (Bronnikov 1979, Bronnikov and Kovalchuk 1980)

$$\begin{aligned} &ds^2 = e^{\kappa p_0 R^2}(dT^2 - dR^2) - R^2\, d\xi^2 - d\eta^2, \\ &\omega = \omega_0 = \tfrac{1}{2}\ln R, \qquad \sigma = \sigma_0 = p_0^{1/2} T, \qquad p = p_0\, e^{-\kappa p_0 R^2}, \end{aligned} \tag{15}$$

with p_0 = constant. One might expect that, when perturbed, such a configuration should radiate. However, it may be shown that perturbations preserving cylindrical symmetry (for which a general solution is obtained) are non-radiative if we require that they are physically meaningful, that is, preserving regularity at the axis $R = 0$ and vanishing for $R \to \infty$.

Physically meaningful, monochromatic, gravitational perturbations with a certain frequency k and an amplitude a_k can be described by the formulae

$$\omega = \omega_0(R) + \omega_k, \qquad \omega_k = a_k J_0(kR)\sin kT, \qquad a_k = \text{constant}, \tag{16}$$

$$\gamma = \gamma_0(R) + \omega_k + \tfrac{1}{2}a_k^2 kR\, J_0(kR) J_1(kR)\cos 2kT, \tag{17}$$

$$2\gamma_0(R) = \kappa p_0 R^2 + k^2 a_k^2 \int dR\, R[J_0^2(kR) + J_1^2(kR)], \tag{18}$$

where J_ν are the Bessel functions. Solutions with the cylindrical functions $N_0(kR)$ and $K_0(kR)$ are ruled out by the regularity condition at the axis; those with $I_0(kR)$ are rejected since they grow exponentially for large R. For monochromatic sound perturbations such that $\sigma_k = \sigma - \sigma_0 = b_k J_0(kR)\sin kT$, one obtains again (17) and (18) up to the changes

$$a_k \to \kappa^{1/2} b_k, \qquad \omega_k \to \kappa^{1/2}\sigma_k. \tag{19}$$

The amplitude b_k is limited by the physical requirement $p \geq 0$. Indeed, the expression

$$p\, e^{2\gamma} = p_0 + 2kp_0^{1/2} b_k J_0 \cos kT + k^2 b_k^2 J_0^2 \cos^2 kT - J_1^2 \sin^2 kT \tag{20}$$

† Scalar fields ψ^A are not included here, otherwise we ought to consider background scalar fields and scalar perturbations, which are, however, decoupled from the ω and σ fields. Consequently, their contribution to the solution is reduced to emergence of new terms in the expression for $\gamma(R, T)$ quite similar to those due to ω or σ. However, the interpretation of the fields ψ^A is quite different, as ω has the geometrical meaning of an anisotropy factor and σ represents the fluid.

(where $J_\nu = J_\nu(kR)$) can change its sign if b_k is too great, and at some values of R and T the pressure and density vanish. Perhaps this should be interpreted as possible stratification of the cylinder under sufficiently large sound perturbations. A similar effect is more important in cosmological-type solutions, see § 4.

As for gravitational perturbations, they preserve the co-moving character of the reference frame and their amplitude a_k is arbitrary, since p obeys the formula $p = p_0 e^{-2\gamma}$ and $\gamma(R, T)$ is finite.

If there is more than one frequency ($\omega = \Sigma\omega_k, \sigma = \Sigma\sigma_k$), the expression for γ involves frequency sums and differences, but it cannot grow with time since there is no zero-frequency term in the expression for $\dot{\gamma}$ in (12).

To reveal the unphysical nature of escaping wave solutions, let us consider, e.g., a monochromatic gravitational wave such that

$$\omega = \omega_0 + a_k J_0(kR) \cos kT + N_0(kR) \sin kT. \tag{21}$$

The functions $\gamma(R, T)$ and $p(R, T)$ are easily found, as described in § 2. For small R their behaviour is dominated by the Neumann function $N_0(kR) \approx (2/\pi) \ln kR$:

$$\gamma \approx [(1 + 2\pi^{-1} a_k \sin kT)^2 - 1] \ln kR, \qquad p = p_0 e^{-2\gamma}. \tag{22}$$

So γ takes infinite values of variable sign and the matter has now zero, now infinite density at the axis. Thus it is not the perturbations of the matter distribution which are the source of the waves, but the pulsating singularity.

For similar sound perturbations (change $\omega \to \sigma$, $a_k \to \kappa^{1/2} b_k$), γ behaves as before, but in the expression for p the main term for small R is negative:

$$p = e^{-2\gamma}(\dot{\sigma}^2 - \sigma'^2) \approx -e^{-2\gamma}(4b_k^2/\pi^2 R^2) \sin^2 kT \leqslant 0. \tag{23}$$

Combinations of waves with different frequencies and phases do not alter the situation.

Thus one should conclude that the configuration (15) is stable under finite cylindrically symmetric perturbations and cannot radiate cylindrical waves of the Einstein–Rosen type (probably it can radiate in more complicated modes, destroying the symmetry).

Arbitrary, physically meaningful, cylindrically symmetric perturbations of ω and σ can be expanded into standing waves, and consequently, cannot propagate from the place where they have emerged†. Such a situation reminds one of Birkhoff's theorem although here a vacuum wave solution exists.

The reason for non-existence of escaping waves (that such waves would violate the regularity condition) seems to be quite general, at least for those equations of state which do not admit matter–vacuum interfaces characterised by $p = 0$ and $\rho \neq 0$. Such interfaces involve discontinuities which, in principle, can allow matching of radiative external solutions to regular internal ones. As for matter whose density vanishes with vanishing pressure, one may suppose that no regular cylindrically symmetric distribution of it can be a source of the Einstein–Rosen waves.

Possibly this conclusion may change if one considers a more general situation, e.g., a cylinder of matter surrounded by radiation with non-zero pressure.

† A superposition of standing waves can, however, describe pulses which come from infinity, are reflected from the axis and escape back, like the example of the Einstein–Rosen waves in a vacuum considered by Weber and Wheeler (1957).

4. Waves in a Kasner-like universe

A homogeneous expanding cosmological model (Bianchi type I) is obtained from § 2 if we put $a^2 = T$, $\omega = \omega_0(T)$ and $\sigma = \sigma_0(T)$:

$$ds^2 = T^{2\bar{b}}(dT^2 - dR^2) - T^{1+b}\, d\xi^2 - T^{1-b}\, d\eta^2,$$

$$\omega = \omega_0 = \tfrac{1}{2}b \ln T, \qquad \sigma = \sigma_0 = p_0^{1/2} \ln T, \qquad p = p_0 T^{-2-2\bar{b}}, \tag{24}$$

$$\bar{b} = \kappa p_0 + (b^2 - 1)/4, \qquad b, p_0 = \text{constant}.$$

The cosmology is isotropic in the special case $b = 0$, $\kappa p_0 = \frac{3}{4}$.

Consider perturbations of this model preserving its pseudoplanar symmetry. For a monochromatic travelling gravitational wave one can write:

$$\omega = \tfrac{1}{2}b \ln T + \omega_k, \qquad \omega_k = a_k(\mathrm{J}_0 \cos kR + N_0 \sin kR), \tag{25}$$

$$\gamma = \bar{b} \ln T + b\omega_k(R, T) + \gamma_k(R, T); \tag{26}$$

$$2a_k^{-2}\gamma_k = kT(N_0N_1 - \mathrm{J}_1\mathrm{J}_0) \cos 2kR - (N_0\mathrm{J}_1 + \mathrm{J}_0N_1) \sin 2kR$$

$$-4kR/\pi + k^2 \int dT\, T(\mathrm{J}_0^2 + \mathrm{J}_1^2 + N_0^2 + N_1^2), \tag{27}$$

where $\mathrm{J}_\nu = \mathrm{J}_\nu(kT)$ and $N_\nu = N_\nu(kT)$. For the asymptotic $kT \gg 1$ certain simplifications occur:

$$\omega_k \approx (2/\pi kT)^{1/2} a_k \cos(kT - kR - \pi/4), \tag{28}$$

$$a_k^{-2}\gamma_k \approx -\cos(2kT - 2kR) + 2k(T - R). \tag{29}$$

For sound waves the same formulae are obtained with the changes $\omega \to \kappa^{1/2}\sigma$, $\omega_k \to \kappa^{1/2}\sigma_k$, $b \to 2p_0^{1/2}\kappa^{1/2}$, $a_k \to \kappa^{1/2}b_k$.

One sees that γ_k contains, along with double harmonic frequency, a term proportional to R which strongly violates the homogeneity of the universe at any moment of time. This violent term appears from a Wronskian of the J and N functions. Thus a travelling wave cannot emerge as a finite perturbation in a homogeneous universe, since it would require an instantaneous global metric change. The same is valid for a wave packet with a spectrum of ω_k, as the sign of the violent term is common for all frequencies and depends only on propagation direction. Frequency sums and differences emerging in $\gamma(R, T)$ do not affect this term. One may conclude that there is a kind of momentum conservation law for the waves: monochromatic waves can exist in pairs propagating in opposite directions with equal ka_k^2. More generally, if the perturbation ω is expanded in ω_k, we should require $\int a_k^2 k\, dk = 0$ if we formally take $k < 0$ for waves travelling in the negative R direction. For joint gravitational and sound perturbations this 'conservation law' is

$$\int (a_k^2 + \kappa b_k^2)k\, dk = 0. \tag{30}$$

It should be added that in model (24) the three Killing directions R, ξ, η are equivalent and one can consider waves propagating in any of them with the same results.

For standing waves the violent term is absent, so they are always allowed as perturbations.

It should be further noted that sound perturbations inevitably cause sign variability in $p = \rho$. Indeed, e.g., for a monochromatic standing wave $\sigma = p_0^{1/2} \ln T +$

$b_k J_0(kT)\cos kR$ one obtains

$$p=(p_0/T^2)-(2p_0^{1/2}/T)kb_k^2 J_1 \cos kR + k^2 b_k^2 (J_1^2 \cos^2 kR - J_0^2 \sin^2 kR). \tag{31}$$

Apparently the third term having a variable sign is the largest in magnitude for large T. (For gravitational perturbations only the first term is present.) However, equations (2) lose their meaning if $p=\sigma^{\alpha}\sigma_{,\alpha}<0$, as the velocity becomes imaginary. Thus the solution is valid up to surfaces where $p\to 0$ and the matter velocity tends to that of light. In the co-moving reference frame, these surfaces appear as repulsive singularities† ($g_{00}\to\infty$, see (A2.3)).

Probably the occurrence of such surfaces means stratification of the homogeneous matter distribution into distinct layers. Moreover, as it has been already mentioned, the three spacial directions in model (24) are essentially equivalent and sound perturbations can arise in any of them, to say nothing of the isotropic case when all the directions are equivalent. It seems reasonable to interpret this behaviour of the perturbations as a mechanism for fragmentation or clustering. A conclusion is that the Kasner-like cosmology is unstable with respect to clustering. The k-dependence of (31) tells us that this process goes faster for shorter wavelengths. An unanswered question remains whether this effect is a special feature of stiff matter models or has a more general significance.

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Appendix 1. Inclusion of electromagnetic plane waves

Consider electromagnetic waves which could propagate in the R direction of space–time (1). In accordance with the symmetry assumed, we take the non-zero components of the 4-potential to be $A_2(R, T)$ and $A_3(R, T)$. The Maxwell equations take the form

$$(e^{-2\omega}\dot{A}_2)^{\cdot}=(e^{-2\omega}A_2')', \qquad (e^{2\omega}\dot{A}_3)^{\cdot}=(e^{2\omega}A_3')', \tag{A1.1}$$

and are easily solved for‡ $\omega=0$ (this condition selects plane-symmetric configurations among the pseudoplanar ones§; for cylindrical and toroidal symmetries $\omega=0$ is just a constraint). Namely,

$$A_a=F_a(T+R)+G_a(T-R), \qquad a=2,3. \tag{A1.2}$$

† The surfaces where $\sigma^{\alpha}\sigma_{,\alpha}=0$ are singularities of just the matter distribution. It is easily verified that there is no space–time singularity and the solution is valid for all R and $T>0$ if σ is interpreted as a common scalar field.

‡ The case $\omega=$ constant is reduced to $\omega=0$ by rescaling $\eta\to$ constant $\times\,\eta$.

§ Existence of non-zero 3-vector quantities in the (ξ,η) plane, the electric and magnetic vectors ($\boldsymbol{E}$ and $\boldsymbol{B}$), does not contradict the space–time planar symmetry. Indeed, this symmetry implies just $T_2^2=T_3^3$, whence $B_2^2+E_2^2=B_3^2+E_3^2$ (where B_a and E_a are the physical magnetic and electric field components in the corresponding direction). Moreover, from the condition $T_{23}=0$ it follows that $B_2B_3=E_2E_3$. Consequently, everywhere the $\boldsymbol{E}$ and $\boldsymbol{B}$ vectors are mutually orthogonal and equal in magnitude. To ensure this, one can, e.g., direct the η axis at an arbitrary point along $\boldsymbol{B}$.

The electromagnetic energy–momentum tensor has the only non-zero components (we put, as before, $\gamma=\lambda$)

$$T^0_0=-T^1_1=(8\pi a^2)^{-1}\,\mathrm{e}^{-2\gamma}(A'_aA'_a+\dot A_a\dot A_a),\qquad T_{01}=(4\pi a^2)^{-1}\dot A_aA'_a. \tag{A1.3}$$

Now suppose that these electromagnetic waves are added to the physical system considered in § 2. One sees that equations (4) $(\binom{0}{0}+\binom{1}{1})$ and (7) (scalar equation) remain unaltered, and in (5) and (6) ((01) and $\binom{1}{1}$, respectively) there appear right-hand sides arising from (A1.3) and consisting of known functions.

Thus, for $\omega=0$, inclusion of electromagnetic waves does not break the solution scheme of § 2, and a general solution to the field equations is obtained in quadratures as well.

Appendix 2. Co-moving frame of reference

To obtain the solution in the co-moving frame which is most suitable for physical interpretation, it is sufficient to transform the results of § 2, putting $T^{\text{comov}}=\sigma(R,T)$ and for cases (i) and (ii) (see (10)):

$$\text{(i)}\quad \mathrm{d}R^{\text{comov}}=(\dot\sigma\,\mathrm{d}R+\sigma'\,\mathrm{d}T)/T,\qquad \text{(ii)}\quad \mathrm{d}R^{\text{comov}}=(\dot\sigma\,\mathrm{d}R+\sigma'\,\mathrm{d}T)/R. \tag{A2.1}$$

However, for the general form of σ it is difficult to find an explicit form of the dependence on T^{comov} and R^{comov}, and it seems reasonable to try to solve the equations anew in the co-moving frame.

If, instead of putting $\gamma=\lambda$, we choose the coordinates R and T as co-moving ones, the set (4)–(7) is modified in the following way: $\omega^A|_{A=2}$ disappears; in (6) there emerges the right-hand side κp; finally, equation (7) for σ is replaced by the 'conservation laws'

$$(a^2\,\mathrm{e}^\lambda p)^{\cdot}+p(a^2\mathrm{e}^\lambda)^{\cdot}=0,\qquad p'+2p\gamma'=0, \tag{A2.2}$$

which give the expressions for p and $\mathrm{e}^{\lambda-\gamma}$:

$$p=p_0\,\mathrm{e}^{-2\gamma}f^2(T),\ p_0=\text{constant};\qquad \mathrm{e}^{\lambda-\gamma}=F(R)/a^2f(T), \tag{A2.3}$$

where the functions f and F may be chosen arbitrarily to concretise the R and T coordinates. Relations (A2.3) decouple the field equations. Indeed, (4) is now a nonlinear equation for $a(R,T)$. It is difficult to solve it completely, but if we manage to find its special solution, we can further (similarly to § 2) find $\omega^A(A\neq 2)$ and γ from the remaining linear equations and λ and p from (A2.3).

In particular, equation (4) has solutions of the form $a^2=U(T)X(R)$. Indeed, choosing $F(R)=X$ and $f(T)=1/U$ (just this choice of F and f gives $\gamma\equiv\lambda$), we obtain the equation

$$X''/X=\ddot U/U=B=\text{constant}. \tag{A2.4}$$

Now (7) is solved by separation of variables: letting $\omega=\sum_k\omega_k$, $\omega_k=\Omega_k(R)w_k(T)$, we obtain

$$(X\Omega')'/X\Omega=(U\dot w)^{\cdot}/Uw=K=\text{constant}. \tag{A2.5}$$

For different variants of solutions of (A2.4), this leads either to equations with constant coefficients, or to Bessel or Legendre equations.

Thus a broad, though not general, class of solutions is obtained directly in the co-moving frame. In particular, for the physical examples of §§ 3 and 4, this leads to a complete treatment of purely gravitational perturbations.

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